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Quantizing strings in de Sitter space

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Abstract: We quantize a string in the de Sitter background, and we find that the mass spectrum is modified by a term which is quadratic in oscillating numbers, and also proportional to the square of the Hubble constant.

Keywords: Bosonic Strings, String theory and cosmic strings.

Contents

1. Introduction

The progress of the string program as a theory of quantum gravity and other interactions is currently impeded by a fundamental difficulty, namely we do not know how to formulate string theory in a time-dependent background in general, and how to understand many issues related to cosmology such as the origin of our universe and the nature of dark energy in particular. This baffling situation leads to lots of debates about whether string theory is the correct theory of nature, and whether string theory has any predictive power if there exists a vast landscape of meta-stable vacua. It goes without saying that string theory has been tremendously successful in resolving some of deeper conceptual problems such as whether gravity is compatible with quantum mechanics, but only in some unrealistic backgrounds such as a flat background and an anti-de Sitter background. In some cases, we even have a non-perturbative formulation, for instance, a CFT is a non-perturbative theory in the AdS/CFT duality. Nevertheless, until we have a theory for time-evolving backgrounds, string theory can not claim to be the theory of our universe.

We shall not try to attack the ultimately difficult problem of formulating string theory in a general or even an “on-shell” time-dependent background in this note. Our purpose is rather pragmatic, we will try to work out part of string quantization in a de Sitter background, with applications to inflation as well as to a later universe dominated by dark energy in mind. For instance, we would like to know how different the spectrum of a string in the de Sitter space is from that in the flat spacetime, whether this spectrum enables string production during inflation. If not, whether strings are created at the end of inflation when the Hubble constant undergoes transition from a constant to a decreasing function. In the later universe such as the current epoch, our universe is again dominated by energy of almost constant density. Although a cosmic string, if exists, is largely governed by classic dynamics, it is certainly of interest to know whether its spectrum is modified in some extremal limit.

The answers to the above questions seem to be yes. As we shall see, the dynamic equation for the field corresponding to a fixed state contains a new term induced by "string mode creation" (to be explained shortly). This term depends on the Hubble constant, thus it renders string creation possible in the end of inflation. This term begins to be comparable to the usual term in the "mass" spectrum when the oscillation numbers are large enough. We put mass into quotation marks since there is no notion of mass in a de Sitter space.

Note that, we will exclusively deal with "small" string states in this paper, by a small string we mean that the string modes are mostly oscillating. It is known that cosmic strings are "long" strings, namely the dominating modes are not oscillating modes so that the major part of the string co-moves with the expansion of the universe.

There is a series of papers on first quantizing string in de Sitter space by de Vega and Sánchez and their collaborators, see for instance [1]–[6]. In their work, they fix all the degrees of freedom of the world sheet metric, and find an exact classical solution. To quantize, they propose two methods, one is to quantize the fluctuation around an exact solution of the center of mass [1], and the other is to propose a quantization condition semiclassically [2].

In this note, we propose a new approach. We leave one degree of the world sheet metric unfixed, and then eliminate it by a constraint. To the leading order, our result agrees with that of de Vega and Sánchez. However, there is a subtle difference: our method is approximate in choosing the gauge. Despite of this, the answer is exact once the gauge is chosen. The most important consequence of our result is that the mode creation and annihilation operators are still time-dependent even after diagonalization. Thus, a state created by these operators is itself time-dependent, and this will have some effects in the dynamic equation for the corresponding field.

We will present our approach in the next section and carry out the first quantization in section 3. We will discuss possible applications to inflation and obtain some conclusions in the last section. A discussion on the mode creation on a string is left in the appendix.

2. Action and gauge choice

We start with the Polyakov action in a general background

$$S = -\frac{1}{4\pi\alpha'}\int d\tau\, d\sigma\, \sqrt{-h}\, h^{ab}\, \partial_a X^\mu\, \partial_b X^\nu\, G_{\mu\nu}(X)\,, \tag{2.1}$$

where $h = \det h_{ab}$ (a, b run over values (τ, σ).), $0 \le \sigma \le 2\pi$, and $G_{\mu\nu}$ is the string frame metric. Here we suppose that the dilaton is constant, and $G_{\mu\nu}$ (μ, ν run over 0, 1, 2, 3.) is the metric of de Sitter space in comoving frame

$$ds^2 = -dt^2 + e^{2Ht}\,(dx^i)^2\,,\ i = 1, 2, 3\ . \tag{2.2}$$

Of course, if we naively take (2.1) as the whole story, then in the background (2.2) a quantum string is not well-defined, since we do not have a two dimensional conformal field theory. We shall assume that there is a hidden sector making the whole world-sheet action conformally invariant (as an example, in a merely AdS space, the bosonic string action is not conformally invariant, and we need another sector from a sphere as well some other terms due to flux and fermionic degrees. As for a de Sitter space, we assume a scenario such as KKLT compactification making the story complete).

By means of the classical world-sheet symmetry, we can set determinant of the world-sheet metric to -1 after choosing the temporal gauge and diagonalizing world-sheet metric. Following this strategy, there is only one component of the world sheet metric $h_{\sigma\sigma}$ left unfixed, which is non-dynamical. We assume that $h_{\sigma\sigma}$ depends only on time, then the target space coordinates X^i can be solved in terms of $h_{\sigma\sigma}$. When we quantize the field X^i, $h_{\sigma\sigma}$ will be promoted to an operator. In order to obtain the on-shell condition, we then impose the constraint from the variation of $h_{\sigma\sigma}$ on physical states. In more detail, the constraint is the integral of the variation of $h_{\sigma\sigma}$ due to our assumption.

Now we perform the steps summarized above. By choosing a proper gauge, we can fix the redundancies in the Polyakov action, and make the equations of motion simple. Set

$$\tau = t\,,\; h^{\tau\sigma} = 0\,,\; -h = 1\ . \tag{2.3}$$

where t is the comoving time. Under this gauge choice, the action becomes

$$S = \frac{1}{4\pi\alpha'}\int dt\, d\sigma\{-h_{\sigma\sigma} + e^{2Ht}\,[h_{\sigma\sigma}\,(\partial_t X^i)^2 - h_{\sigma\sigma}^{-1}\,(\partial_\sigma X^i)^2]\}\ . \tag{2.4}$$

There are two independent constraints due to functional variation of h^{ab}, that is,

$$\frac{1}{4\pi\alpha'}\,e^{2Ht}\,\partial_t X^i(t,\sigma)\,\partial_\sigma X^i(t,\sigma) = 0\,, \tag{2.5}$$

$$\frac{1}{8\pi\alpha'}\{-h_{\sigma\sigma}^2(t,\sigma) + e^{2Ht}\,[h_{\sigma\sigma}^2(t,\sigma)(\partial_t X^i(t,\sigma))^2 + (\partial_\sigma X^i(t,\sigma))^2]\} = 0\ . \tag{2.6}$$

The equations of motion corresponding to the functional variation of X^μ are

$$\partial_t(e^{2Ht}\,h_{\sigma\sigma}(t,\sigma)\,\partial_t X^i(t,\sigma)) - e^{2Ht}\,\partial_\sigma(h_{\sigma\sigma}^{-1}(t,\sigma)\,\partial_\sigma X^i(t,\sigma)) = 0\,, \tag{2.7}$$

$$\partial_t h_{\sigma\sigma}(t,\sigma) + H\,e^{2Ht}\,[h_{\sigma\sigma}(t,\sigma)(\partial_t X^i(t,\sigma))^2 - h_{\sigma\sigma}^{-1}(t,\sigma)(\partial_\sigma X^i(t,\sigma))^2] = 0\ . \tag{2.8}$$

The second equation of motion (2.8) comes from the functional variation of X^0, which is non-dynamical according to our gauge choice. This becomes another constraint, but fortunately it can be derived from the other three equations. So at last we get two constraints and three equations of motion (each for one spatial coordinate).

The conjugate momentum of $X^i(t,\sigma)$ is given by $\Pi_i(t,\sigma) = \frac{1}{2\pi\alpha'}\,e^{2Ht}\,h_{\sigma\sigma}(t,\sigma)\,\partial_t X^i(t,\sigma)$. The Hamiltonian is then

$$E = \frac{e^{2Ht}}{4\pi\alpha'}\int d\sigma\,[\frac{h_{\sigma\sigma}(t,\sigma)}{e^{2Ht}} + h_{\sigma\sigma}(t,\sigma)(\partial_t X^i(t,\sigma))^2 + h_{\sigma\sigma}^{-1}(t,\sigma)\,(\partial_\sigma X^i(t,\sigma))^2] \tag{2.9}$$

$$\simeq \frac{1}{2\pi\alpha'}\int d\sigma\, h_{\sigma\sigma}(t,\sigma)\ . \tag{2.10}$$

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In the last step we used (2.6), which is satisfied only by physical states. We use the symbol $\simeq$ instead of $=$ to show that the equality is satisfied only by physical states. Hence $\frac{1}{2\pi\alpha'}\int d\sigma h_{\sigma\sigma}$ is in fact just the energy of a physical state with respect to comoving time. For simplicity, we will set $\alpha' = 1$ hereafter. $h_{\sigma\sigma}$ is non-dynamical, and is determined by $X^i(t,\sigma)$ through (2.6). If we eliminate $h_{\sigma\sigma}$, (2.7) will become nonlinear equations which is hard to solve. Instead, we will first treat $h_{\sigma\sigma}$ as an independent variable, solve the equations of motion for $X^i(t,\sigma)$ in terms of $h_{\sigma\sigma}$ and then fix $h_{\sigma\sigma}$ by (2.6). In order to solve the equation of motion (2.7), we make an assumption that

$$h_{\sigma\sigma}(t,\sigma) = \omega(t). \tag{2.11}$$

This is the only approximation we execute in this note. For strings oscillating fast, $\omega(t)$ may be viewed as an average of $h_{\sigma\sigma}(\sigma,t)$ along σ.

Hereafter we will just write ω instead of $\omega(t)$ for simplicity, but keep in mind that ω is in fact a function of time. Also, we caution that upon quantization, X^i become operators, so does ω, namely ω is not to be viewed as a usual function.

For physical states, $E \simeq \omega$ is the energy of the string, the equations of motion become

$$\partial_t(\eta^{-2}\,\partial_t X^i(t,\sigma)) - \omega^{-2}\,\eta^{-2}\,\partial_\sigma^2 X^i(t,\sigma) = 0\,, \tag{2.12}$$

where $\eta = \frac{1}{e^{Ht}\sqrt{\omega}}$. A general solution is

$$\begin{aligned} X^i(t,\sigma) &= x_0 + \int^t du\,\eta^2(u)\,p^i + \sum_{m\in Z/\{0\}} \eta(t)\Big[\frac{a^i_m(t)}{\sqrt{2\,|\lambda_m(t)|}}\,e^{-i\int^t du\,\lambda_m(u)}\,e^{im\sigma} \\ &\quad + \frac{\tilde{a}^i_m(t)}{\sqrt{2|\lambda_m(t)|}}\,e^{-i\int^t du\,\lambda_m(u)}\,e^{-im\sigma}\Big]\,, \end{aligned} \tag{2.13}$$

$$\dot{a}^i_m(t) = \frac{\dot{\lambda}_m(t)}{2\lambda_m(t)}\,\tilde{a}^i_{-m}(t)\,e^{2i\int^t du\lambda_m(u)},\ \dot{\tilde{a}}^i_m(t) = \frac{\dot{\lambda}_m(t)}{2\lambda_m(t)}\,a^i_{-m}(t)\,e^{2i\int^t du\lambda_m(u)}\,. \tag{2.14}$$

where we have defined$\lambda_m = sgn(m)\sqrt{\frac{m^2}{\omega^2} - \eta\partial_t^2(\eta^{-1})} = sgn(m)\sqrt{\frac{m^2}{\omega^2} - (H+\frac{\dot{\omega}}{2\omega})^2 - \partial_t(\frac{\dot{\omega}}{2\omega})}$, where the function $sgn(m) = 1$ for $m > 0$, and $sgn(m) = -1$ for $m < 0$. We will work in the situation that λ_ms remain real, which means that the string is oscillating in time. This is to be dubbed as a small string, since it is not stretched too much with the expansion of the universe.

The real condition for λ_m is that $\frac{m^2}{\omega^2} - (H+\frac{\dot{\omega}}{2\omega})^2 - \partial_t(\frac{\dot{\omega}}{2\omega}) > 0$. The Hermiticy of $X^i(t,\sigma)$ requires that $(a^i_m)^\dagger = a^i_{-m}$ and $(\tilde{a}^i_m)^\dagger = \tilde{a}^i_{-m}$. Thus the conjugate momentum of X^i becomes

$$\begin{aligned} \Pi^i(t,\sigma) &= \frac{1}{2\pi}\{p^i + \sum_{m\in Z/\{0\}}[\frac{\dot{\eta}(t)}{\eta(t)^2} - \frac{i\lambda_m(t)}{\eta(t)}]\,[\frac{a^i_m(t)}{\sqrt{2\,|\lambda_m(t)|}}\,e^{-i\int^t du\,\lambda_m(u)}\,e^{im\sigma} \\ &\quad + \frac{\tilde{a}^i_m(t)}{\sqrt{2|\lambda_m(t)|}}\,e^{-i\int^t du\,\lambda_m(u)}\,e^{-im\sigma}]\}\,. \end{aligned}$$

To quantize, impose the equal time canonical commutation relations

$$[X^i(t,\sigma), X^j(t,\sigma')] = [\Pi_i(t,\sigma), \Pi_j(t,\sigma')] = 0\,, \tag{2.15}$$

$$[X^i(t,\sigma), \Pi_j(t,\sigma')] = i\,\delta^i_j\,\delta(\sigma-\sigma')\ . \tag{2.16}$$

which are equivalent to imposing the commutation relations

$$[x^i, x^j] = [p^i, p^j] = 0\ ,\ [x^i, p^j] = i\delta^{ij} \tag{2.17}$$

$$[a^i_m, \tilde{a}^j_n] = 0\ ,\ [a^i_m, a^j_n] = \frac{m}{|m|}\delta^{ij}\delta_{m,-n}\,. \tag{2.18}$$

Note that a^i_m and $\tilde{a}^i_m$ depend on ω implicitly. Thus when we impose (2.18), we have also promote ω to be an operator, which commutes with other operators including $\dot{\omega}$. To simplify the notations, we will not distinguish operators and functions explicitly except that ω, p and N are assumed to be operators when constructing states.

3. Quantization

We now study constraints (2.5), (2.6) and (2.8). (2.8) can be derived from (2.5), (2.6) together with the equation of motion (2.7). Thus only (2.5) and (2.6) need to be considered. According to our assumption $h_{\sigma\sigma}(t,\sigma) = \omega(t)$, the most important parts of these constraints are their average over σ. Thus the constrains become,

$$\begin{aligned} P &\equiv \int \frac{d\sigma}{4\pi}\, e^{2Ht}\, \partial_t X^i(t,\sigma)\, \partial_\sigma X^i(t,\sigma) \\ &= \sum_{m>0} \frac{m}{2\omega}\, \{a^i_{-m}\, a^i_m - \tilde{a}^i_{-m}\, \tilde{a}^i_m\} \simeq 0\,, \end{aligned} \tag{3.1}$$

and

$$\mathscr{H} \equiv \int \frac{d\sigma}{8\pi}\, \{h^2_{\sigma\sigma}\,[-1 + e^{2Ht}\,(\partial_t X^i(t,\sigma))^2] + e^{2Ht}\,(\partial_\sigma X^i(t,\sigma))^2\} \tag{3.2}$$

$$\begin{aligned} &= -\frac{\omega^2}{4} + \frac{(p^i)^2}{4\,e^{2Ht}} + \sum_{m>0,i} \frac{\omega}{4\lambda_m}\,[(\frac{\dot{\eta}}{\eta})^2 + \lambda_m^2 + \frac{m^2}{\omega^2}]\,(a^i_{-m}\, a^i_m + \tilde{a}^i_{-m}\, \tilde{a}^i_m + 1) \\ &+ \frac{\omega}{4\lambda_m} e^{-2i\int^t du \lambda_m(u)}\,[(\frac{\dot{\eta}}{\eta} - i\lambda_m)^2 + \frac{m^2}{\omega^2}]\, a^i_m\, \tilde{a}^i_m \\ &+ \frac{\omega}{4\lambda_m} e^{2i\int^t du \lambda_m(u)}\,[(\frac{\dot{\eta}}{\eta} + i\lambda_m)^2 + \frac{m^2}{\omega^2}]\, a^i_{-m}\, \tilde{a}^i_{-m} \simeq 0\ . \end{aligned} \tag{3.3}$$

There are also infinitely many constraints corresponding to positve modes of (2.5), (2.6) in the Fourier expansion in terms of σ. Note that our temporal gauge choice only eliminate the temporal degrees of freedom. These infinitely many constraints, whose analog in flat spacetime are the conditions $L_n = \tilde{L}_n = 0$, will further eliminate the unphysical degrees of freedom due to the longitudinal excitations. In this note we will not discuss them in detail, and will just focus on the mass shell condition, whose analog in flat spacetime is $L_0 = 0$. One may ask that why there are still two set of constraints while only the longitudinal

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degrees of freedom need to be eliminated. Are our constraints too strong? The answer is no, because we have used the assumption that $h_{\sigma\sigma}$ is independent of σ, which imposes another constraint. If we re-introduce $h_{\sigma\sigma}$ as an arbitrary function of σ, the counting of degrees of freedom will be correct.

Define the occupation number $n^i_m = a^i_{-m}\, a^i_m$, and the level $n = \sum_{m,i} m\, n^i_m$, and similarly for $\tilde{n}$. Then the vanishing of (3.1) on physical states is just the level matching condition $n = \tilde{n}$. Note that the condition (3.1) is also the translational invariance condition along σ. We can make a linear transformation to define another set of creation and annihilation operators,

$$A^i_m = \alpha_m\, a^i_m + \beta_m\, \tilde{a}^i_{-m},\ \tilde{A}^i = \tilde{\alpha}_m\, \tilde{a}^i_m + \tilde{\beta}_m\, a^i_{-m}\ , \tag{3.4}$$

$$|\alpha_m|^2 - |\beta_m|^2 = |\tilde{\alpha}_m|^2 - |\tilde{\beta}_m|^2 = 1\ , \tag{3.5}$$

$$\alpha_m \beta_{-m} = \tilde{\alpha}_m \tilde{\beta}_{-m}\ . \tag{3.6}$$

The conditions (3.5) and (3.6) ensure that the new operators satisfy the same commutation relation as (2.18). The above transformaion may be viewed as Bogoliubov transformation on the world-sheet. The most general form is

$$\alpha_m = \cosh(\gamma_m)\, e^{i\delta_m + i\phi_m}, \tilde{\alpha}_m = \cosh(\gamma_m)\, e^{i\delta_m + i\psi_m}, \tag{3.7}$$

$$\beta_m = \sinh(\gamma_m)\, e^{i\phi_m}, \quad \tilde{\beta}_m = \sinh(\gamma_m)\, e^{i\psi_m}\ . \tag{3.8}$$

where γ_m, ϕ_m, ψ_m and δ_m are real. Constraint (3.1) remains the level matching condition, but now in terms of the new occupation number operator $N = \tilde{N}$, defined by $N = \sum_{i,m} m N^i_m$, and $N^i_m \equiv A^i_{-m}\, A^i_m$, and similarly for $\tilde{N}$.

We now choose the parameters γ_m and δ_m properly to diagonalize the constraint (3.2). The conditions are

$$\cosh^2(\gamma_m) + \sinh^2(\gamma_m) = \frac{\omega}{2m\,\lambda_m}\,[(\frac{\dot{\eta}}{\eta})^2 + \lambda_m^2 + \frac{m^2}{\omega^2}]\ , \tag{3.9}$$

$$\sinh(2\gamma_m) = \frac{\omega}{2m\,\lambda_m}\,[(\frac{\dot{\eta}}{\eta} - i\lambda_m)^2 + \frac{m^2}{\omega^2}]\, e^{-2i\int^t du\, \lambda_m(u) - i\delta_m} \tag{3.10}$$

where δ_m is chosen to make $\sinh(2\gamma_m)$ real, and this can be done.

In terms of the new creation and annihilation operators

$$\begin{aligned} \mathscr{H} &= -\frac{\omega^2}{4} + \frac{(p^i)^2}{4e^{2Ht}} + \sum_{m>0,i} \frac{m}{2}(N^i_m + \tilde{N}^i_m + 1) \\ &\simeq -\frac{\omega^2}{4} + \frac{(p^i)^2}{4e^{2Ht}} + N + \frac{E_0}{2} \simeq 0\,,\ E_0 = -\frac{1}{4}\ . \end{aligned} \tag{3.11}$$

The fact that we need to form a Bogoliubov transformation implies that if we start with a state constructed by the original operators a^i_m, there will be mode creation along the string at a later time.

We now discuss implication for dynamics of fields viewed as coefficients in the expansion of a general string state.

Since ∂_i is a Killing vector, p^i is conserved, and we will just work in momentum representation. Any physical state should satisfy the constraint $\mathscr{H} = 0$, and can be expanded in terms of common eigenstates of the occupation number operator N_m^is, $\tilde{N}_m^i$s, p^i as well as ω. That is,

$$|\phi>= \sum_{N_1^1,\tilde{N}_1^1,......N_m^i,\tilde{N}_m^i,......} |N_1^1,\tilde{N}_1^1,......N_m^i,\tilde{N}_m^i,.......\omega,p^i> \phi(N_m^i,\tilde{N}_m^i,\omega,p^i)\ , \quad (3.12)$$

where the eigenvalues labeling the states must satisfy the relation $-\omega^2+\frac{(p^i)^2}{e^{2Ht}}+2(2N+E_0)=0$, and $N=\tilde{N}$.

When writing down the action for the above general state, it is important to keep in mind that the inner product involves an integral over space so there is a nontrivial Hermticity condition. For instance, we consider a scalar particle with mass m, whose wave function must satisfy

$$(\Box - m^2)\phi(x) = [-\partial_t^2 - 3H\partial_t + e^{-2Ht}(\partial^i)^2 - m^2]\phi(x) = 0\ . \quad (3.13)$$

Written in momentum representation, the last two terms are $-e^{-2Ht}\ (p^i)^2 - m^2 = -E^2$, where E is the comoving energy.

In string field theory, we assume that the action of string state has the form of

$$S = \int dt < \phi|\partial_t^2 + 3H\partial_t + E^2 + \lambda\mathscr{H}|\phi> \quad (3.14)$$

where λ is the Lagrangian multiplier. In writing down this action, we have only considered the mass shell condition and have omitted other constraints corresponding to the positive Fourier modes. To treat the problem more rigorously, one should introduce more Lagrangian multipliers. From this action, we can easily get the evolution equation of the string state. By variation of $|\phi>$, we have

$$(\partial_t^2 + 3H\partial_t + E^2)|\phi>= [\partial_t^2 + 3H\partial_t + e^{-2Ht}\ (p^i)^2 + 2(N+\tilde{N}+E_0)]|\phi>= 0\ . \quad (3.15)$$

This equation contains an unphysical component which is to be discarded due to the fact that in the action the inner production automatically projects out the unphysical component by imposing the constraint $\mathscr{H}|\phi>= 0$.

We explain some subtleties in deriving this second order equation. Because of the non-flat metric, the measure of the integral volume is $d\vec{x}^3\sqrt{-G} = d\vec{x}^3 e^{3Ht}$, so the inner product should be defined as $\int d^3x e^{3Ht}\phi(x)^*\phi(x)$. With this inner product, ∂_t^2 is not Hermitian. To get a Hermitian operator, we should replace ∂_t^2 with $\partial_t^2 + 3H\partial_t$. There is no addition term caused by polarization indices since all the creation operators are properly normalized.

Now, different excitation modes of the string correspond to different particles in space-time, thus the coefficient $\phi(N_m^i,\tilde{N}_m^i,\omega,p^i)$ is the wave function of single particle. From now on, we will use notation $|N_m^i,\tilde{N}_m^i,\omega,p^i>$ instead of $|N_1^1,\tilde{N}_1^1,......N_m^i,\tilde{N}_m^i,.......\omega,p^i>$ for simplicity. Since the basis $|N_m^i,\tilde{N}_m^i,\omega,p^i>$ evolves with time, this dependence on time will be transmitted into the equation of motion for ϕ through the action of $\partial_t^2 + 3H\partial_t$.

Using the differential equation for a^i_m and $\tilde{a}^i_m$, (2.14), and the definition of A^i_m, $\tilde{A}^i_m$, we have

$$
\begin{aligned}
\dot{A}^i{}_{-m} &= c_m\,\tilde{A}^i_m + d_m\,A^i{}_{-m},\ \dot{\tilde{A}}^i{}_{-m} = \tilde{c}_m\,A^i_m + \tilde{d}_m\,\tilde{A}^i{}_{-m}\,, \qquad (3.16)\\
c_m &= e^{i(\phi-\psi)}\left\{\alpha^*_m\dot{\beta}^*_m - \beta^*_m\dot{\alpha}^*_m + [\alpha^{*2}e^{-2i\int^t du\lambda_m(u)} - \beta^{*2}e^{2i\int^t du\lambda_m(u)}]\frac{\dot{\lambda}_m}{2\lambda_m}\right\}\\
&= e^{(-i\delta-i\phi-i\psi)}H\frac{\partial_t\frac{\dot{\omega}}{2\omega} - i\frac{2m}{\omega}(H+\frac{\dot{\omega}}{2\omega})}{\sqrt{(\frac{2m}{\omega})^2(H+\frac{\dot{\omega}}{2\omega})^2 + [\partial_t(\frac{\dot{\omega}}{2\omega})]^2}}\,, \qquad (3.17)\\
\tilde{c}_m &= e^{i(\psi-\phi)}\left\{\tilde{\alpha}^*_m\dot{\tilde{\beta}}^*_m - \tilde{\beta}^*_m\dot{\tilde{\alpha}}^*_m + [(\tilde{\alpha}^*)^2e^{-2i\int^t du\lambda_m(u)} - (\tilde{\beta}^*)^2e^{2i\int^t du\lambda_m(u)}]\frac{\dot{\lambda}_m}{2\lambda_m}\right\}\\
&= e^{(-i\delta-i\phi-i\psi)}H\frac{\partial_t\frac{\dot{\omega}}{2\omega} - i\frac{2m}{\omega}(H+\frac{\dot{\omega}}{2\omega})}{\sqrt{(\frac{2m}{\omega})^2(H+\frac{\dot{\omega}}{2\omega})^2 + [\partial_t(\frac{\dot{\omega}}{2\omega})]^2}}\,, \qquad (3.18)\\
d_m &= e^{i(\phi-\psi)}\left\{\tilde{\alpha}_m\dot{\alpha}^*_m - \tilde{\beta}_m\dot{\beta}^*_m + \frac{\dot{\lambda}_m}{2\lambda_m}[\tilde{\alpha}_m\beta^*_m e^{2i\int^t du\,\lambda_m(u)} - \alpha^*_m\tilde{\beta}_m e^{-2i\int^t du\,\lambda_m(u)}]\right\}\\
&= i\frac{H+\frac{\dot{\omega}}{2\omega}}{2\lambda_m^2(\frac{2m^2}{\omega^2} - \partial_t\frac{\dot{\omega}}{2\omega} - \frac{2m}{\omega}\lambda_m)}\left\{\frac{4Hm}{\omega}\lambda_m^2 + \lambda_m\left[\partial_t^2\frac{\dot{\omega}}{2\omega} + 2\left(H+\frac{\dot{\omega}}{2\omega}\right)\partial_t\frac{\dot{\omega}}{2\omega} + \frac{4m^2}{\omega^2}\frac{\dot{\omega}}{2\omega}\right]\right\}\\
&\quad -i\dot{\phi}\,, \qquad (3.19)\\
\tilde{d}_m &= e^{i(\psi-\phi)}\left\{\alpha_m\dot{\tilde{\alpha}}^*_m - \beta_m\dot{\tilde{\beta}}^*_m + \frac{\dot{\lambda}_m}{2\lambda_m}[\alpha_m\tilde{\beta}^*_m e^{2i\int^t du\,\lambda_m(u)} - \tilde{\alpha}^*_m\beta_m e^{-2i\int^t du\,\lambda_m(u)}]\right\}\\
&= i\frac{H+\frac{\dot{\omega}}{2\omega}}{2\lambda_m^2(\frac{2m^2}{\omega^2} - \partial_t\frac{\dot{\omega}}{2\omega} - \frac{2m}{\omega}\lambda_m)}\left\{\frac{4Hm}{\omega}\lambda_m^2 + \lambda_m\left[\partial_t^2\frac{\dot{\omega}}{2\omega} + 2\left(H+\frac{\dot{\omega}}{2\omega}\right)\partial_t\frac{\dot{\omega}}{2\omega} + \frac{4m^2}{\omega^2}\frac{\dot{\omega}}{2\omega}\right]\right\}\\
&\quad -i\dot{\psi}\ .
\end{aligned}
$$

Note that $|c_m| = H$. We can choose ϕ and ψ to set $d_m = \tilde{d}_m = 0$, and then the phase of c_m is also fixed.

The eigenstates of N^i_m and $\tilde{N}^i_m$ are just $\Pi_{m,i}(A^i{}_{-m})^{N^i_m}(\tilde{A}^i{}_{-m})^{\tilde{N}^i_m}\,|\Omega,\omega>$, where $|\Omega,\omega>$ is defined as $A^i_m|\Omega,\omega>=0$, $\tilde{A}^i_m|\Omega,\omega>=0$ for all m,i. From $\dot{A}^i_m\,|\Omega,\omega> + A^i_m\,\partial_t|\Omega,\omega>=0$, we get

$$\partial_t|\Omega,\omega>= -\sum_{m,i} c^*_m\,A^i{}_{-m}\,\tilde{A}^i{}_{-m}|\Omega,\omega>\ , \qquad (3.20)$$

and

$$
\begin{aligned}
&\partial_t|N^k_n,\tilde{N}^k_n,\omega>=\sum_{m,i}[c_m\,A^i_m\,\tilde{A}^i_m - c^*_m\,A^i{}_{-m}\,\tilde{A}^i{}_{-m}]|N^k_n,\tilde{N}^k_n,\omega>\ , \qquad (3.21)\\
&\partial_t^2|N^k_n,\tilde{N}^k_n,\omega>= -\sum_{m,i}|c_m|^2\,(1+2N^i_m\,\tilde{N}^i_m + N^i_m + \tilde{N}^i_m)|N^k_n,\tilde{N}^k_n,\omega>\ , \qquad (3.22)\\
&+\{-\sum_{m,i,l,j} 2\,c_m\,c^*_l\,A^i{}_{-l}\,\tilde{A}^i{}_{-l}\,A^j_m\,\tilde{A}^j_m + \sum_{m,i}\dot{c}_m\,A^i_m\,\tilde{A}^i_m - \sum_{m,i}\dot{c}^*_m\,A^i{}_{-m}\,\tilde{A}^i{}_{-m} \qquad (3.23)\\
&+\sum_{m,i,l,j}[c_l\,c_m\,A^i_l\,\tilde{A}^i_l\,A^j_m\,\tilde{A}^j_m + c^*_l\,c^*_m\,A^i{}_{-l}\,\tilde{A}^i{}_{-l}\,A^j{}_{-m}\,\tilde{A}^j{}_{-m}]\}|N^k_n,\tilde{N}^k_n,\omega>\,, \qquad (3.24)
\end{aligned}
$$

$$\dot{\omega}|N_n^k, \tilde{N}_n^k, \omega >= -\frac{Hp^2 e^{-2Ht}}{\omega}|N_n^k, \tilde{N}_n^k, \omega > , \tag{3.25}$$

$$\dot{N} + \dot{\tilde{N}} = \sum_{m,i} 2m(c_m\, A_m^i\, \tilde{A}_m^i + c_m^*\, A_{-m}^i\, \tilde{A}_{-m}^i) \ . \tag{3.26}$$

From the explicit expression above, we see that the off diagonal parts of $\partial_t|N, \tilde{N}, \omega >$ and $\partial_t^2|N_m^i, \tilde{N}_m^i, \omega >$ are unphysical if $|N_m^i, \tilde{N}_m^i, \omega >$ is physical. So the off diagonal part is orthogonal to physical states.

Just considering the physical part of the following equation

$$\{\partial_t^2 + 3H\partial_t + (p^i)^2\, e^{-2Ht} + 4N + 2E_0\}|\phi >= 0\,,$$

we have

$$\begin{aligned} &\{\partial_t^2 + 3H\partial_t + (p^i)^2\, e^{-2Ht} + 4N + 2E_0 \\ &-\sum_{m,i} H^2\,(1 + 2N_m^i\,\tilde{N}_m^i + N_m^i + \tilde{N}_m^i)\}\phi(N_n^k, \tilde{N}_n^k, \vec{p}) = 0 \ . \end{aligned} \tag{3.27}$$

The above equation is the main result of this note. We see that in addition to the term $4N$, there is an additional term which is quadratic in creation numbers with a prefactor H^2. This term could become comparable to the linear term N for a fixed H^2 (measured in the string unit since we have set $\alpha' = 1$). Restoring the string scale M_s, we find that the new term is comparable to the old term $M_s^2 N$ when $N \sim M_s^2/H^2$. This is of course a large number during inflation because H is much smaller than the string scale, if we hope that an effective field theory is valid.

This quadratic term is negative when viewed as a contribution to the mass squared m^2, thus it appears possible to have an effective negative mass squared if the quadratic term becomes dominating. This will never become possible, since in our approach so far we have assumed real λ_m in the mode expansion (2.14), and we are dealing with "small" strings mostly oscillating in time. It can be checked that as long as the real condition on λ_m is met, the quadratic term in (3.27) will never make the mass squared negative. Nevertheless, the fact that this term reduces the mass squared comes as a surprise. We may just imagine that this is a quantum effect for a highly excited state due to the mode creation on the string. We leave a discussion on mode creation to the appendix.

4. Comparison with earlier results

The spectrum we get is

$$\alpha' M^2 = 4N + 2E_0 - \sum_{m,i} H^2\,(1 + 2N_m^i\,\tilde{N}_m^i + N_m^i + \tilde{N}_m^i), \tag{4.1}$$

which can be read from (3.27). The previous result obtained by the method of [1] is

$$\alpha' M^2 = 24 \sum_{n>0} \frac{2n^2 - H^2 M^2 \alpha'^2}{\sqrt{n^2 - H^2 M^2 \alpha'^2}} + 2N \frac{2 - H^2 M^2 \alpha'^2}{\sqrt{1 - H^2 M^2 \alpha'^2}}\,, \tag{4.2}$$

where we cite the formula in the form appearing in [2], and the dimensionality of the spacetime is 25 there. We can see that when $H = 0$, both results return to the flat spacetime spectrum, $4N+$zero point energy. The difference between the zero point energy is due to the different dimensionality. When $H \neq 0$, both results have a term proportional to H^2, but the coefficients are different. What's more, our result shows that the spectrum depends not only on the level N, but also on the specific excitation. This difference may due to the gauge choice of the two approaches. As we have mentioned before, we fix the worldsheet time τ, set the determinant of the worldsheet metric to be one, set one component of the worldsheet metric $h^{12} = 0$, and leave another worldsheet metric component $h^{\sigma\sigma}$ unfixed. To set the determinant of the worldsheet metric to be one, we have used the classical conformal symmetry of the action, which does not exist in the whole theory. While in the earlier approach, for example, [1], they choose the conformal gauge and leave the worldsheet coordinate τ unfixed. Since there is no conformal symmetry, there is no guarantee that we should get the same result. The other approach previously developed in [2], namely, the semiclassical quantization, also take the same conformal gauge, and make a circular string ansatz, and then they propose a quantization condition. The spectrum is $\alpha' m^2 \approx 5.9n, n \in N_0$ in [2]. In [2], the authors compare this result with that obtained in [1] by calculating the maximum excitation number of string states in de Sitter spacetime. In [1], the maximum number of a single excitation is $N_{\max} = \frac{0.15}{H^2\alpha'}$, which obtained in [2] is $N_{\max} = \frac{0.17}{H^2\alpha'}$. In our approach, the condition for the state to oscillate is $\lambda_m^2 \geq 0$. Together with (3.12), we will get roughly $N_{\max} = \frac{0.25}{H^2\alpha'}$ for all the oscillating modes to have real frequency. Our result is slightly larger than the previous results. One possibility is that our solution is more general than the previous ones. For instance, in [2], only circular solutions are considered, and in [1], only expansion around an exact solution is considered. While here in our paper, only one approximation is made, namely, the worldsheet metric $h^{\sigma\sigma}$ does not depend on σ. Thus we might have found more solutions.

5. Discussions and conclusion

In this paper, we first quantized a general oscillating string in a de Sitter space , with the only approximation that $h_{\sigma\sigma}$ depends only on time. This quantity becomes the energy density along the string after we impose constraint condition on it. So our approximation amounts to averaging the energy density along the string. Aside from this, our treatment is exact.

Apparently, our main result (3.27) differs from the old result by a negative contribution to the mass squared of the string. This term which is quadratic in oscillation numbers increases quickly when we consider more highly excited states. Applying this result to inflation, there is virtually no physical effect during inflation except for modification of the mass spectrum, for this new term depends only on the Hubble constant thus remains a constant for a given state. It is a well-known result that for a particle of constant mass, there is no particle production during inflation. Although we have worked with a constant Hubble constant, some of our results can be generalized to a non-constant Hubble constant. We expect string creation in the end of inflation can be induced by this new term , for the

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Hubble constant is no longer a constant in this short reheating period . A similar effect is discussed in [7] and [8] except for ad hoc coupling to some moduli. We leave a detailed investigation of string creation to a future work [9].

In this paper we have restricted attention to “small” string states, namely strings with oscillating modes only. It can be expected that the phenomenon of string mode creation and its induced effects on the equation of motion prevails for the long strings which stretch with the expansion of the universe. Again, detailed result will be presented in [9].

We also expect the new term we discovered will have some effects for cosmic strings at later times.

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A. Mode creation

To investigate the question of mode creation, we need to consider the expectation value of the occupation number operator on a state that does not change with time, that is, the eigenstates of the time independent number operator. Because $\dot{A}^i_{\pm m}$ and $\dot{\tilde{A}}^i_{\pm m}$ do not vanish, the corresponding number operators $\hat{N}$ and $\hat{\tilde{N}}$ depend on time. Thus we need to find a set of creation and annihilation operators b^i_m and $\tilde{b}^i_m$, s.t. $\dot{b}^i_{\pm m} = \dot{\tilde{b}}^i_{\pm m} = 0$. $b^i_{\pm m}$ and $\tilde{b}^i_{\pm m}$ are linear combinations of $A^i_{\pm m}$ and $\tilde{A}^i_{\pm m}$. As we have mentioned under (3.4), the most general form of linear transformation preserving the commutation relation is to set

$$b^i_m = \cosh(\gamma'_m) e^{i\phi'_m + i\delta'_m} A^i_m + \sinh(\gamma'_m) e^{i\phi'_m} \tilde{A}^i_{-m} \ , \tag{A.1}$$

$$\tilde{b}^i_m = \cosh(\gamma'_m) e^{i\psi'_m + i\delta'_m} \tilde{A}^i_m + \sinh(\gamma'_m) e^{i\psi'_m} A^i_{-m} \ . \tag{A.2}$$

Demanding that

$$\begin{aligned} \dot{b}^i_m = & [i(\dot{\phi}'_m + \dot{\delta}'_m)\cosh(\gamma'_m)e^{i\phi'_m + i\delta'_m} + \dot{\gamma}'_m \sinh(\gamma'_m) e^{i\phi'_m + i\delta'_m} + c_m \sinh(\gamma'_m) e^{i\phi'_m}] A^i_m \\ & + [i\dot{\phi}'_m \sinh(\gamma'_m) e^{i\phi'_m} + \dot{\gamma}'_m \cosh(\gamma'_m) e^{i\phi'_m} + c^*_m \cosh(\gamma'_m) e^{i\phi'_m + i\delta'_m}] \tilde{A}^i_{-m} = 0 \ , \end{aligned} \tag{A.3}$$

$$\begin{aligned} \dot{\tilde{b}}^i_m = & [i(\dot{\psi}'_m + \dot{\delta}'_m)\cosh(\gamma'_m)e^{i\psi'_m + i\delta'_m} + \dot{\gamma}'_m \sinh(\gamma'_m) e^{i\psi'_m + i\delta'_m} + c_m \sinh(\gamma'_m) e^{i\psi'_m}] \tilde{A}^i_m \\ & + [i\dot{\psi}'_m \sinh(\gamma'_m) e^{i\psi'_m} + \dot{\gamma}'_m \cosh(\gamma'_m) e^{i\psi'_m} + c^*_m \cosh(\gamma'_m) e^{i\psi'_m + i\delta'_m}] A^i_{-m} = 0 \ . \end{aligned} \tag{A.4}$$

Then we have

$$\dot{\gamma}'_m + \Re(c_m e^{-i\delta'_m}) = 0 \ , \tag{A.5}$$

$$\dot{\delta}'_m + [\tanh(\gamma'_m) + \coth(\gamma'_m)] \Im(c_m e^{-i\delta'_m}) = 0 \ . \tag{A.6}$$

Suppose that at time $t = t_0$, the two sets of operators are identical, e.g. $b^i_m = A^i_m(t_0)$, $\tilde{b}^i_m = \tilde{A}^i_m(t_0)$. Then the Hilbert space spanned by eigenstates of $N' \equiv b^i_{-m} b^i_m$ and $\tilde{N}' \equiv \tilde{b}^i_{-m} \tilde{b}^i_m$ is identical with that of $\hat{N}(t_0)$ and $\hat{N}(t_0)$. Thus we have the initial condition $\gamma'_m = \delta'_m = 0$.

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Denote $|t_o>$ a state that does not change with time, then $< t_0|\hat{N}(t)|t_0 >$ represents the change of the total mode number.

$$< t_0|\dot{N} + \dot{\tilde{N}}|t_0 >= -\sum_{m,i} 2mH \sinh(2\gamma'_m)\cos(\delta'_m)(N^i_m + \tilde{N}^i_m + 1) \ . \tag{A.7}$$

When $\vec{p} = 0$, both the real part and the imaginary part of c_m is pure oscillating, so the average of $\dot{\gamma}'_m$ and $\dot{\delta}'_m$ vanish, with the initial condition, we will have $\gamma'_m \approx \delta'_m \approx 0$. Then $< t_0|\dot{N}|t_0 >\approx 0$.

References

[1] H.J. de Vega and N.G. Sanchez, *A new approach to string quantization in curved space-times*, *Phys. Lett.* **B 197** (1987) 320.

[2] H.J. de Vega, A.L. Larsen and N.G. Sanchez, *Semiclassical quantization of circular strings in de Sitter and anti-de Sitter space-times*, *Phys. Rev.* **D 51** (1995) 6917 [hep-th/9410219].

[3] A.L. Larsen and N.G. Sánchez, *Mass spectrum of strings in anti-de Sitter space-time*, *Phys. Rev.* **D 52** (1995) 1051 [hep-th/9410132].

[4] M. Ramon Medrano and N.G. Sanchez, *QFT, string temperature and the string phase of de Sitter space time*, *Phys. Rev.* **D 60** (1999) 125014 [hep-th/9904015].

[5] P. Bozhilov, *Exact string solutions in nontrivial backgrounds*, *Phys. Rev.* **D 65** (2002) 026004 [hep-th/0103154].

[6] A. Bouchareb, M. Ramon Medrano and N.G. Sánchez, *Semiclassical (quantum field theory) and quantum (string) de Sitter regimes: new results*, hep-th/0511281.

[7] S.S. Gubser, *String production at the level of effective field theory*, *Phys. Rev.* **D 69** (2004) 123507 [hep-th/0305099].

[8] S.S. Gubser, *String creation and cosmology*, hep-th/0312321.

[9] C.J. Feng, X. Gao, M. Li, W. Song and Y. Song, work in progress.